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Perturbation Theory For Implosions

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PERTURBATION THEORY FOR IMPLOSIONS

by

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Perturbation Theory for Implosions

Cathleen S. Morawetz

1. Introduction.

If a piston is pushed into a closed tube of gas a shock is created which races ahead of the piston to the end of the tube. There it is reflected and returns back to the piston. This process is very well-known and the flow has been completely determined for arbitrary piston motions. In this paper we are concerned with an analogous problem in three dimensions, the so-called implosion phenomenon. A spherical shell, or "piston", is contracted toward its center, thereby creating a shock which precedes it into the center. The problem is to determine the flow inside the shell and to decide whether the shock is reflected out from the center.

Guderley [1] has determined such a flow for a special choice of piston velocity. In this particular case, the shock is reflected at the center and expands back toward the shell just as in the one-dimensional case. One very interesting result is that the pressure behind the reflected wave is about twenty-six times the incident pressure while in the one-dimensional case the ratio is only eight. We shall call this flow the basic flow.

The question arises immediately as to whether this flow pattern is general or just a special effect, resulting from the choice of piston velocity, that happens to resemble the

one-dimensional case. We would like in fact to show that the flow resulting from an arbitrary piston motion behaves like the basic flow near the center and that therefore the reflection pattern at the center is essentially the same in all cases.

By using the method of perturbation it is possible to show that if the piston is moved with a velocity somewhat different from that which yields the basic flow the difference between the two flows disappears at the center. Thus it is to be expected that pistons contracting with arbitrary velocity produce flows with reflection properties similar to the basic flow.

In addition it is shown how to determine the perturbation terms that are due to a shift in the time scale or in the distance scale, see sections 13 and 14 of Part III.

The basic flow is the particular flow inside a contracting spherical piston which can be described by

$$u = \lambda^{-1} r t^{-1} U_o(r^{-\lambda} t)$$

$$c = \lambda^{-1} r t^{-1} C_o(r^{-\lambda} t)$$

$$p = \lambda^{-2} r^2 t^{-2} P_o(r^{-\lambda} t)$$

$$z = \lambda^{-1} r t^{-1}.$$

Here u, c, p and z are the velocity, sound speed, pressure and shock velocity, t is time and r is the distance from the center. U_o, C_o and P_o are all functions of

one variable, $\eta = r^{-\lambda}t$, which are chosen so that the differential equations of motion and the shock conditions are satisfied. $\lambda = 1.3947$ is a fixed constant and cannot be chosen arbitrarily as we shall see later.

We shall consider flows produced by an appropriate piston motion which can be described by

$$u = \lambda^{-1} r t^{-1} (U_0(\eta) + \varepsilon U_1(\eta, r))$$

$$c = \lambda^{-1} r t^{-1} C_0(\eta)(1 + \varepsilon C_1(\eta, r))$$

$$p = \lambda^{-2} r^2 t^{-2} P_0(\eta)(1 + \varepsilon P_1(\eta, r))$$

with $\eta = r^{-\lambda}t$. Here ε is small and terms of the order ε^2 are neglected. In this case a preliminary but lengthy calculation indicates that if we keep η constant and let $r \longrightarrow 0$ we find that the quantities U_1, C_1 and P_1 all tend to zero, approximately like r^2.

At some future date, it is intended that these calculations will be extended in a more rigorous way.

Thus the flow behaves like the basic flow near the center. Furthermore, the singularities that occur at the center of the flow will be the same as those at the center of the basic flow. Finally, the shock will be reflected at the center and large pressures, similar to those in the basic flow, will occur.

Part I. Differential Equations and Boundary Conditions.

2 Formulation.

We assume that the spherical piston is contracting with

velocity u_p on the enclosed gas which is originally in a constant state at rest, that is, it has velocity $u_o = 0$, pressure p_o = constant and density ρ_o = constant. The velocity u, pressure p, and density ρ resulting in the gas satisfy the equations for spherically symmetric motion, (see [2], p. 418),

$$
\begin{aligned}
& u_t + u\, u_r + \frac{1}{\rho} p_r = 0 \\
(1) \qquad & \rho_t + u\, \rho_r + \rho(u_r + \frac{2u}{r}) = 0 \\
& (p\rho^{-\gamma})_t + u(p\rho^{-\gamma})_r = 0.
\end{aligned}
$$

It is assumed here that the gas is polytropic.

The given path of the piston is described by

$$(2) \qquad r = R_p(t)$$

and at the shock, the conditions,

$$
\begin{aligned}
& \rho(u - z) = -\rho_o\, z \\
(3) \qquad & \rho(u - z)^2 + p = \rho_o\, z^2 + p_o \\
& \mu^2(u - z)^2 + (1 - \mu^2)c^2 = \mu^2\, z^2 + (1 - \mu^2)c_o^2
\end{aligned}
$$

hold where z is the shock velocity, $c = \sqrt{\frac{\gamma p}{\rho}}$ is sound speed and $\mu^2 = \frac{\gamma - 1}{\gamma + 1}$.

Guderley has found that if appropriate new variables are introduced special solutions of (1) can be found by solving

one non-linear differential equation of first order. One such solution represents a flow between a contracting shock and a reflected expanding shock. We use the same change of variables. We introduce instead of t the independent variable

$$(4) \qquad \eta = r^{-\lambda} t$$

and the dependent variables

$$(5) \qquad \begin{aligned} U &= \lambda t r^{-1}\, u \\ C &= \lambda t r^{-1}\, c \\ \Omega &= \rho \\ P &= \lambda^2 t^2 r^{-2}\, p \end{aligned}$$

where λ is an appropriate positive constant.

From (4) we obtain

$$(6) \qquad \frac{d\eta}{\eta} = -\lambda \frac{dr}{r} + \frac{dt}{t} .$$

Using (4), (5) and (6) the equations of motion become, with $a = \lambda^{-1}$,

$$(7) \qquad \eta U_\eta (1 - U) + aUrU_r - \gamma^{-1} C^2 \frac{\eta P_\eta}{P} + a\gamma^{-1} C^2 \frac{rP_r}{P} + U(aU-1) + 2a\gamma^{-1} C^2 = 0$$

$$-\eta U_\eta + arU_r + \frac{\eta P_\eta}{P} (1 - U) - 2 \frac{\eta C_\eta}{C} (1 - U) + aU \frac{rP_r}{P} - 2aU \frac{rC_r}{C} + 3aU = 0$$

$$\frac{\eta P_\eta}{P}(1 - U) - \frac{2\gamma}{\gamma-1}(1 - U)\frac{\eta C_\eta}{C} + \alpha U\frac{rP_r}{P} - \frac{2\alpha\gamma}{\gamma-1}U\frac{rC_r}{C} + 2\,\frac{(1-\alpha U)}{\gamma-1} = 0$$

where $\Omega = \gamma P/C^2$.

These equations may be rewritten in the form,

$$D\eta U_\eta = A - \alpha(C^2 + U(1-U))rU_r - \alpha\gamma^{-1}C^2\frac{rP_r}{P}$$

(8) $$D\frac{\eta C_\eta}{C} = B - \alpha\frac{\gamma-1}{2}rU_r - \frac{\alpha UD}{1-U}\frac{rC_r}{C} - \frac{\alpha(\gamma-1)}{2\gamma}\frac{C^2}{(1-U)}\frac{rP_r}{P}$$

$$D\frac{\eta P_\eta}{P} = E - \alpha\gamma rU_r - \alpha(C^2 + U(1-U))\frac{rP_r}{P}$$

where

$$A = (1 - U)(1 - \alpha U)U - (3\alpha U - 2\gamma^{-1}(1 - \alpha))C^2$$

(9) $$B = (1 - U)(1 - \alpha U) - (\gamma - 1)[(1 - \alpha U) - \tfrac{3}{2}(1 - \alpha)]U - [\alpha + \gamma^{-1}(1 - \alpha)(1 - U)^{-1}]C^2$$

$$E = (1 - U)[2(1 + \gamma)(1 - \alpha U) - 3\gamma] + \gamma(1 - \alpha U) - 2\alpha C^2$$

$$D = (1 - U)^2 - C^2.$$

For further reference we note here the identities

(10) $$-A + \frac{2}{\gamma-1}B(1 - U) + D[3\alpha U - \frac{2}{\gamma-1}(1 - \alpha U)] = 0$$

$$-A + \gamma^{-1}E(1 - U) + D[3\alpha U - \frac{2}{\gamma}(1 - \alpha U)] = 0.$$

At the piston we have

$$U = U_p = \lambda \eta r^{\lambda-1} \frac{d}{dt} R_p \tag{11}$$

and immediately behind the shock

$$\begin{aligned} &\Omega(U - Z) = -\rho_o Z \\ &\Omega(U - Z)^2 + P = \rho_o Z^2 + \lambda^2 \eta^2\, r^{2\lambda-2} p_o \\ &\mu^2(U - Z)^2 + (1 - \mu^2)C^2 = \mu^2 Z^2 + (1 - \mu^2)\lambda^2 \eta^2 r^{2\lambda-2} c_o{}^2 \end{aligned} \tag{12}$$

where $\Omega = \gamma P/C^2$, see (7), and

$$Z = \lambda t r^{-1} z. \tag{13}$$

3. The Basic Solution.

Guderley's solutions are those solutions of (8) and (12) which are independent of r. In such a solution the position of the shock is constant, say $\eta = r^{-\lambda} t = H_o$ and the shock velocity is therefore given by $z = ar/t$. From (13) it follows that $Z = 1$. This is compatible with (12) only if we have $p_o = c_o = 0$ or, in other words, if the shock is infinitely strong.

From (8), we find for this case

$$\begin{aligned} D\eta U_\eta &= A \\ D\eta C_\eta &= BC \\ D\eta P_\eta &= EP \end{aligned} \tag{14}$$

and from (12),

$$\begin{aligned} U &= 1 - \mu^2 \\ C &= \pm\mu\sqrt{1 + \mu^2} \\ P &= (1 - \mu^2)\rho_o. \end{aligned} \tag{15}$$

Since the piston is contracting, the gas moves towards the center, i.e., u is negative at the junction of the piston and the shock. From (4), (5) and (14) we see that $t < 0$, $\eta < 0$, corresponds to $u < 0$. Since $c = \sqrt{\frac{\gamma p}{\rho}} > 0$ we have from (5) that $t < 0$ corresponds to $C < 0$. For a contracting piston we must therefore choose the negative sign in the second equation of (14).

In order that equations (14) have a solution satisfying the initial conditions (15), that does not become singular for $D = 0$, the value of α must be specially chosen. The correct value for α is approximately .717 with $\gamma = 1.4$. In this case then $A = B = E = 0$ when $D = 0$ but the ratios A/D, B/D, and E/D remain finite.

To the incident flow satisfying (14) and (15) with $\alpha = .717$ a reflected shock can be fitted. The flow behind this reflected shock is easily determined. In the incident flow U lies between $(1 - \mu^2)$ and some negative value behind the reflected shock while C lies between $-\mu\sqrt{1 + \mu^2}$ and some positive value.

The picture of the flow is then as follows: Up to some time, $\bar{t}_0$ say, the piston is fixed at $r = \bar{R}_0$ where $\bar{R}_0^{-\lambda}\,\bar{t}_0 = H_0$ and the state in front of it is constant. Then for $t \geq \bar{t}_0$ the position of the piston, using (5) and (6), is given by

$$(16)\qquad r = R_0(\eta) = \bar{R}_0 \exp \int_{H_0}^{\eta} \frac{\alpha U_0}{1-U_0} \frac{d\eta}{\eta}$$

where $U_0(\eta)$ is the solution of (14) and (15) with $\alpha = .717$. The state in front of the piston is still constant for $\eta < H_0$. For $\eta > H_0$ and up to $\eta = H_{00}$ which represents the position of the reflected shock the state is given by the solution $U_0(\eta)$, $C_0(\eta)$, $P_0(\eta)$ of (14) and (15). The particle paths in this region are given by the equation

$$(17)\qquad \log r = \int^{\eta} \frac{\alpha U_0}{1-U_0} \frac{d\eta}{\eta} + \text{constant}.$$

The characteristics in the (r,t)-plane are given by

$$(18)\qquad \frac{dr}{dt} = (u \pm c),$$

see [2], p. 46, and are therefore given in the (η,r)-plane by

$$(19)\qquad \frac{dr}{r\,d\eta} = \frac{1}{(1-U_0) \pm C_0}\,.$$

4. Perturbation.

The present treatment consists in perturbing about Guderley's solution and studying the behavior of the first order terms.

We assume once more that the state on which the piston

contracts has zero pressure[*],

$$(20) \qquad p_o = c_o = 0.$$

Up to time $t = \bar{t}_o + \varepsilon \bar{t}_1$, suppose the piston is fixed at the position $r = \bar{R}_o + \varepsilon \bar{R}_1$ and at $t = \bar{t}_o + \varepsilon \bar{t}_1$, it starts moving on a path given by

$$(21) \qquad r = R_p(\eta) = R_o(\eta) + \varepsilon\, R_1(\eta)$$

where

$$(22) \qquad R_o(\eta) = \bar{R}_o \exp \int_{H_o}^{\eta} \frac{\alpha U_o}{1-U_o} \frac{d\eta}{\eta} \quad \text{(Guderley path).}$$

We are evidently free to choose $\bar{t}_1$ and $\bar{R}_1 = R_1(H_o)$ since any change in these quantities can be absorbed into $\bar{R}_o$ and $\bar{t}_o$.

It is reasonable to expect then that all the flow quantities can be expanded in powers of ε, that is,

$$(23) \qquad \begin{aligned} U &= U_o(\eta) + \varepsilon\, U_1(\eta, r) \ldots \\ C &= C_o(\eta)(1 + \varepsilon C_1(\eta, r) \ldots) \\ P &= P_o(\eta)(1 + \varepsilon P_1(\eta, r) \ldots) \end{aligned}$$

where $U_o(\eta)$, $C_o(\eta)$, $P_o(\eta)$ is Guderley's solution of (14) and (15) in Section 3. Substituting (23) in (8) and taking first order terms in ε we find:

---

[1] It can be shown that if p_o is considered small the resulting disturbance dies out at the center like $r^{2\lambda-2}$, cf.[3].

$$\eta U_{1\eta} = (\tfrac{A}{D})^o_U\, U_1 + (\tfrac{A}{D})^o_C\, C_oC_1 - D_o^{-1}(C_o^2 + U_o(1-U_o))arU_{1r}$$
$$- D_o^{-1}\gamma^{-1}C_o^2 arP_{1r}$$

(24) $$\eta C_{1\eta} = (\tfrac{B}{D})^o_U\, U_1 + (\tfrac{B}{D})^o_C\, C_oC_1 - \frac{\gamma-1}{2} D_o^{-1}arU_{1r} - \frac{U_o}{1-U_o}\, arC_{1r}$$
$$- \frac{\gamma-1}{2\gamma}\, \frac{C_o^2}{1-U_o}\, D_o^{-1}arP_{1r}$$

$$\eta P_{1\eta} = (\tfrac{E}{D})^o_U\, U_1 + (\tfrac{E}{D})^o_C\, C_oC_1 - \gamma D_o^{-1}arU_{1r}$$
$$- D_o^{-1}(C_o^2 + U_o(1-U_o))arP_{1r}.$$

Here $(\frac{A}{D})^o_U = \left[\frac{\partial}{\partial U}\,(\frac{A}{D})\right]_{U = U_o(\eta),\ C = C_o(\eta)}$

$(\frac{A}{D})^o_C = \left[\frac{\partial}{\partial C}\,(\frac{A}{D})\right]_{U = U_o(\eta),\ C = C_o(\eta)}$, etc.

From equations (10) we see that

(25) $$(\frac{2}{\gamma-1})(\tfrac{B}{D})_C - (\tfrac{A}{D})_C\, \frac{1}{1-U} = 0$$
$$\gamma^{-1}\ (\tfrac{E}{D})_C - (\tfrac{A}{D})_C\, \frac{1}{(1-U)} = 0.$$

Using (25) with (24), we obtain

$$\frac{2}{\gamma-1}\,(\eta C_{1\eta}) - \frac{1}{1-U_o}(\eta U_{1\eta}) = \left(\frac{2}{\gamma-1}\,(\tfrac{B}{D})^o_U - \frac{1}{1-U_o}\,(\tfrac{A}{D})^o_U\right)U_1$$
$$- \frac{arU_{1r}}{1-U_o} - \frac{U_o}{1-U_o}\,\frac{2}{\gamma-1}\, arC_{1r}$$

(26) $$\gamma^{-1}(\eta P_{1\eta}) - \frac{1}{1-U_o}\,(\eta U_{1\eta}) = \left(\gamma^{-1}(\tfrac{E}{D})^o_U - \frac{1}{1-U_o}\,(\tfrac{A}{D})^o_U\right)U_1$$
$$- \frac{arU_{1r}}{1-U_o} - \frac{\gamma^{-1}U_o}{1-U_o}\, arP_{1r}$$

$$\eta U_{1\eta} = (\tfrac{A}{D})^o_U\, U_1 + (\tfrac{A}{D})^o_C\, C_oC_1 - (C_o^2 + U_o(1-U_o))D_o^{-1}arU_{1r}$$
$$- \gamma^{-1}C_o^2D_o^{-1}\ arP_{1r}.$$

The form of these equations suggests changing to new dependent variables

$$\begin{aligned} V_1(\eta,r) &= -\frac{U_1(\eta,r)}{U_o(\eta)(1-U_o(\eta))} \\ V_2(\eta,r) &= \gamma^{-1}P_1(\eta,r) + V_1(\eta,r)U_o(\eta) \qquad (27) \\ V_3(\eta,r) &= \frac{2}{\gamma-1}C_1(\eta,r) + V_1(\eta,r)U_o(\eta). \end{aligned}$$

Equations (26) then reduce to

$$\begin{aligned} \eta V_{1\eta} = \frac{\alpha U_o}{1-U_o}\Bigg\{ &-\left(1+\frac{2C_o^2}{U_oD_o}\right) rV_{1r} + \frac{C_o^2}{U_o^2D_o} rV_{2r} + \\ &\lambda V_1\left[\frac{2U_o-1}{U_o^2}\,\frac{A_o}{D_o} + \frac{1-U_o}{U_o}\left(\frac{A}{D}\right)_U^o + \frac{\gamma-1}{2}\,\frac{C_o}{U_o}\left(\frac{A}{D}\right)_C^o\right] - \left(\frac{A}{D}\right)_C \frac{C_o}{U_o^2}\,\frac{\gamma-1}{2}\,\lambda V_3\Bigg\} \end{aligned}$$

$$\begin{aligned} (28) \quad \eta V_{2\eta} &= \frac{\alpha U_o}{1-U_o}\left\{rV_{1r} - rV_{2r} + \lambda V_1[3\alpha-2\gamma^{-1}(1-\alpha)]\right\} \\ \eta V_{3\eta} &= \frac{\alpha U_o}{1-U_o}\left\{rV_{1r} - rV_{3r} + \lambda V_1\left[3\alpha - \frac{2}{\gamma-1}(1-\alpha)\right]\right\}. \end{aligned}$$

We introduce the new independent variables

$$\begin{aligned} y &= \int_{H_*}^{\eta} \frac{\alpha U_o(\eta)}{1-U_o(\eta)}\,\frac{d\eta}{\eta} \\ (29) \qquad x &= \int_{H_*}^{\eta} \frac{\alpha U_o(\eta)}{1-U_o(\eta)}\,\frac{d\eta}{\eta} - \log r = y - \log r \end{aligned}$$

in which H_* is the value of η for which $1 - U_o + C_o = 0$, that is, $D_o(H_*) = 0$. Note that the lines x = constant are the particle paths of Guderley's flow, see (17), and that $y = 0$ is a Mach line, see (19). Increasing t, which corresponded to increasing η, now corresponds to decreasing y. With these variables equations (28) become

$$D_oV_{1y} - \frac{2C_o^2}{U_o} V_{1x} + \frac{C_o^2}{U_o^2} V_{2x} + V_1K_o + V_3L_o = 0$$

$$(30) \qquad V_{2y} + V_{1x} = -\mu_2V_1$$

$$V_{3y} + V_{1x} = -\mu_3V_1$$

where

$$K_o = K_o(y) = -\lambda\left\{\frac{2U_o-1}{U_o^2} A_o + \frac{1-U_o}{U_o} D_o\left(\frac{A}{D}\right)^o_U + \frac{\gamma-1}{2} \frac{C_o}{U_o} D_o\left(\frac{A}{D}\right)^o_C\right\}$$

(31)

$$L_o = L_o(y) = \lambda \frac{\gamma-1}{2} \frac{C_o}{U_o^2} D_o\left(\frac{A}{D}\right)^o_C , \quad \mu_2 = 3 - 2\gamma^{-1}\lambda + 2\gamma^{-1},$$

$$\mu_3 = 3 - \frac{2}{\gamma-1} \lambda + \frac{2}{\gamma-1} .$$

These are the differential equations that are to be solved.

5. Boundary Conditions at the Shock.

We assume the path of the shock is represented by

$$(32) \qquad \eta = H_o(1 + \varepsilon H_1(r) + \ldots)$$

$$\text{or} \quad t = r^\lambda H_o(1 + \varepsilon H_1(r) + \ldots) \quad .$$

The piston starts to move at time $\bar{t}_o + \varepsilon\bar{t}_1$ from the position $r = \bar{R}_o + \varepsilon\bar{R}_1$. We assume that the shock issues from the piston at the same time, that is,

$$(33) \quad \bar{t}_o + \varepsilon\bar{t}_1 = (\bar{R}_o + \varepsilon\bar{R}_1)^\lambda H_o(1 + \varepsilon H_1(\bar{R}_o + \varepsilon\bar{R}_1) + \ldots).$$

On taking first order terms in ε, we obtain

$$H_1(\bar{R}_o) = \frac{\bar{t}_1}{H_o\bar{R}_o} - \alpha\bar{R}_1 \bar{R}_o^{\lambda-2}$$

or using the zero order result, $\bar{t}_o = \bar{R}_o^{\lambda} H_o$,

$$(34) \qquad H_1(\bar{R}_o) = \bar{R}_o^{\lambda-1}\left[\frac{\bar{t}_1}{\bar{t}_o} - \frac{\alpha\bar{R}_1}{\bar{R}_o}\right].$$

In this relation, see Section 4, $\bar{t}_1$ and $\bar{R}_1$ can be chosen arbitrarily.

Along the shock path we have

$$(35) \qquad \frac{dr}{dt} = z = \alpha\,\frac{r}{t}\,Z \quad \text{or} \quad Z = \lambda\ d\ \log r/d\ \log t$$

and hence, by (32),

$$(36) \qquad Z = \frac{1}{1 + \dfrac{\alpha\varepsilon r H_{1r}}{1+\varepsilon H_1}} = 1 - \varepsilon\alpha r H_{1r} + \ldots$$

where $H_1(r) = dH_1/dr$. Substituting (36) in (12) and using (15) and (20) we find on the shock,

$$(37) \qquad \begin{aligned} U &= U(H_o(1+\varepsilon H_1(r),r) = (1-\mu^2)(1-\varepsilon\alpha r H_{1r}\ldots) \\ C &= C(H_o(1+\varepsilon H_1(r),r) = \mu\sqrt{1+\mu^2}\ (1-\varepsilon\alpha r H_{1r}\ldots) \\ P &= P(H_o(1+\varepsilon H_1(r),r) = (1-\mu^2)\rho_o(1-2\varepsilon\alpha r H_{1r}\ldots) \end{aligned}$$

$\Omega = \dfrac{\gamma P}{C^2}$, see (7).

Expanding U, C, P, in powers of ε as functions of ε for fixed η, r we find using (14)

$$(38) \qquad \begin{aligned} U(H_o(1+\varepsilon H_1(r)),r,\varepsilon) &= U_o(H_o) + \varepsilon U_1(H_o,r) + \varepsilon H_1(r)\left(\frac{\partial U}{\partial\eta}\right)_{\varepsilon=0} \\ &= U_o(H_o) + \varepsilon U_1(H_o,r) + \varepsilon\left(\frac{A_o}{D_o}\right)_{\eta=H_o} H_1(r)\ldots \\ C(H_o(1+\varepsilon H_1(r)),r,\varepsilon) &= C_o(H_o)\left[1+\varepsilon C_1(H_o,r)+\varepsilon\ \frac{B_o}{D_o}_{\eta=H_o}\ H_1(r)\ldots\right] \end{aligned}$$

$$P(H_o(1+\varepsilon H_1(r)),r,\varepsilon) = P_o(H_o)\left[1+\varepsilon P_1(H_o,r)+\varepsilon\left(\frac{E_o}{D_o}\right)_{\eta=H_o} H_1(r)\ldots\right].$$

By comparing (37) and (38), and using (15), we obtain the boundary conditions for U_1, C_1, P_1 at the undisturbed shock curve $\eta = H_o$,

$$\begin{aligned}
U_1(H_o,r) &= -\left(\frac{A_o}{D_o}\right)_{\eta=H_o} H_1(r) - (1-\mu^2)arH_{1r}(r) \\
C_1(H_o,r) &= -\left(\frac{B_o}{D_o}\right)_{\eta=H_o} H_1(r) - (1-\mu^2)arH_{1r}(r) \qquad (39) \\
P_1(H_o,r) &= -\left(\frac{E_o}{D_o}\right)_{\eta=H_o} H_1(r) - 2arH_{1r}(r).
\end{aligned}$$

We suppose now that the undisturbed shock curve $\eta = H_o$ corresponds to $y = y(H_o) = Y_o$. The three shock conditions (39) expressed in terms of the variables V_1, V_2, V_3, x and y, see (27) and (29), are then

$$\begin{aligned}
V_1(x,Y_o) &= p_1H(x) + q_1H_x(x) \\
V_2(x,Y_o) &= p_2H(x) + q_2H_x(x) \qquad (40) \\
V_3(x,Y_o) &= p_3H(x)
\end{aligned}$$

where

$$H(x) = H_1(r) \qquad (41)$$

$$p_1 = (1-\mu^2)^{-1}\mu^{-2}\left(\frac{A_o}{D_o}\right)_{\eta=H_o} \qquad (42)$$

$$p_2 = \left(-\gamma^{-1}\frac{E_o}{D_o} + \frac{1}{\mu^2}\frac{A_o}{D_o}\right)_{\eta=H_o}$$

$$p_3 = \left(- \frac{2}{\gamma - 1} \frac{B_o}{D_o} + \frac{1}{\mu^2} \frac{A_o}{D_o} \right)_{\eta = H_o}$$

$$q_1 = -\alpha/\mu^2$$

$$q_2 = (2\gamma^{-1} - \mu^{-2} + 1)\alpha.$$

These boundary conditions contain an undetermined function $H(x)$ which can however be eliminated in an obvious way. The equations which result can be written in the form:

$$p_3 q_2 V_1(x, Y_o) - p_3 q_1 V_2(x, Y_o) + (p_2 q_1 - p_1 q_2) V_3(x, Y_o) = 0$$

(43)

$$p_2 p_3 V_1(x, Y_o) - p_1 p_3 V_2(x, Y_o) + (p_1 q_2 - p_2 q_1) V_{3x}(x, Y_o) = 0$$

with the additional condition $V_3(X_o, Y_o) = p_3 \bar{R}_o^{\lambda - 1} \left[\frac{\bar{t}_1}{\bar{t}_o} - \alpha \frac{\bar{R}_1}{R_o} \right]$ by (34), (40) and (42).

6. Boundary Conditions at the Piston.

We assume that the piston velocity is given through equations (21) by

(44) $$U = U_p(\eta) = U_o(\eta) + \varepsilon U_{1p}(\eta).$$

The connection between U_o, U_{1p}, R_o and R_1 will be given later. However from (23) we have

$$U(\eta, R_o(\eta) + \varepsilon R_1(\eta), \varepsilon) = U_o(\eta) + \varepsilon R_1(\eta) \frac{\partial}{\partial r}(U_o(\eta)) + \varepsilon U_1(\eta, R_1)$$

$$= U_o(\eta) + \varepsilon U_1(\eta, R_o(\eta))$$

or using (44)

(46) $$U_1(\eta, R_o(\eta)) = U_{1p}(\eta).$$

In the (x,y)-plane the undisturbed piston path $r = R_o(\eta)$ is given according to (22) and (29) by the line x = constant, say,

(47) $$x = X_o.$$

Expressing (46) in terms of the variables x, y, and V, by (27) and (29) we have

(48) $$V_1(X_o, y) = - \frac{U_{1p}(\eta(y))}{U_o(1-U_o)} .$$

We next determine $U_{1p}(\eta)$ in terms of $U_o(\eta)$, $R_o(\eta)$ and $R_1(\eta, r)$. From $r = R_p(r^{-\lambda}t)$, see (2), we have

$$\frac{dr}{dt} = \frac{\eta}{r} \frac{dR_p}{d\eta} \left(-\lambda \frac{dr}{dt} + \frac{r}{t}\right) ;$$

hence we obtain using $U = \frac{\lambda t}{r} \frac{dr}{dt}$;

(49) $$\frac{dR_p}{d\eta} = \frac{\alpha R_p(\eta)}{\eta} \frac{U_p(\eta)}{1-U_p(\eta)} .$$

From $r = R_p(\eta) = R_o(\eta) + \varepsilon R_1(\eta)$, $U = U_p(\eta) = U_o(\eta) + \varepsilon U_{1p}(\eta)$ we obtain to first order

$$\frac{dR_o}{d\eta} + \varepsilon \frac{dR_1}{d\eta} = \frac{\alpha(R_o(\eta)+\varepsilon R_1(\eta))}{\eta} \left[\frac{1}{1-U_o(\eta)}\right] \left[1 + \frac{\varepsilon U_{1p}(\eta)}{1-U_o(\eta)}\right] \left[U_o(\eta)+\varepsilon U_{1p}(\eta)\right]$$

$$= \frac{\alpha R_o}{\eta} \frac{U_o}{1-U_o} + \frac{\alpha\varepsilon}{\eta(1-U_o)} \left[U_o R_1 + \frac{R_o U_o}{1-U_o} U_{1p} + U_o R_o U_{1p}\right]$$

or

$$\eta \frac{dR_1}{d\eta} = \frac{\alpha U_o}{1-U_o} R_1 + \frac{U_{1p}}{(1-U_o)^2} \alpha R_o$$

or

$$(50) \qquad U_{1p}(\eta) = \frac{\lambda(1-U_o)^2}{R_o}\left[\eta \frac{dR_1}{d\eta} + R_1\eta \frac{dR_o}{d\eta}\right].$$

7. Conditions at the Junction of Shock and Piston.

From (34) and (41) we have the value of $H(x)$ for $x = X_o$,

$$(51) \qquad H(X_o) = \overline{R}_o^{\,\lambda-1}\left[\frac{\overline{t}_1}{\overline{t}_o} - a\,\frac{\overline{R}_1}{\overline{R}_o}\right].$$

From the first shock condition (40) and equations (21), (48) and (50) we obtain a relation between $H_x(X_o)$ and $H(X_o)$,

$$p_1H(X_o) + q_1H_x(X_o) = -\frac{\lambda(1-U_o(H_o))}{U_o(H_o)R_o}\left[H_o\left(\frac{dR_1}{d\eta}\right)_{\eta=H_o} + \overline{R}_1H_o\left(\frac{dR_o}{d\eta}\right)_{H_o}\right]$$

or using (22) and (29)

$$(52) \qquad p_1H(X_o) + q_1H_x(X_o) = \left(\frac{dR_1}{dy}\right)_{y=Y_o} - \overline{R}_1$$

where we now consider R_1 as a function of y instead of η, see (29).

8. The function $G(x,y)$.

The problem has now been reduced to the following: the three functions V_1, V_2 and V_3 satisfy three homogeneous linear partial differential equations (30) in x and y whose coefficients depend on only one variable y and are analytic. One boundary condition (48) is given on the piston path $x = X_o$ and two conditions (43) on the shock path $y = Y_o$. Both these lines are timelike. We may expect that there exists a unique analytic solution of this problem if the boundary

conditions are analytic. Once its solution is known the function $H(x)$ can be found from the last equation of (40).

The form of equations (30) leads us to introduce the function $G(x,y)$ such that,

(53) $$V_1 = G_y$$

(54) $$G(x,Y_o) = 0.$$

Then we may integrate the last two equations of (30) and obtain, by (54),

(55) $$V_2(x,y) = V_2(x,Y_o) - G_x(x,y) + \mu_2 G(x,y)$$

(56) $$V_3(x,y) = V_3(x,Y_o) - G_x(x,y) + \mu_3 G(x,y).$$

Substituting (53), (55) and (56) into the first equation of (30) and using the shock conditions (40) we find

(57) $$D_o G_{yy} = \frac{2C_o^2}{U_o} G_{xy} + \frac{C_o^2}{U_o^2} G_{xx} - K_o G_y + \left(L_o - \mu_2 \frac{C_o^2}{U_o^2}\right) G_x - \mu_3 L_o G$$
$$- p_3 L_o H(x) - p_2 \frac{C_o^2}{U_o^2} H_x(x) - q_2 \frac{C_o^2}{U_o^2} H_{xx}(x).$$

In this equation, we recall D_o, U_o, C_o, K_o and L_o are all functions of y. On multiplying through by the quantity $\frac{y}{D_o}$ which has a finite value at $y = 0$, $D_o = 0$, we obtain finally

(58) $$yG_{yy} + a(y)G_y + b(y)G_{xy} + c(y)G + d(y)G_x + e(y)G_{xx}$$
$$= \ell(y)H(x) + m(y)H_x(x) + n(y)H_{xx}(x).$$

with

$$
\begin{aligned}
a(y) &= \frac{y}{D_o} K_o \\
b(y) &= -\frac{y}{D_o} \frac{2C_o^2}{U_o} \\
c(y) &= \frac{y}{D_o} \mu_3 L_o \\
d(y) &= -\frac{y}{D_o}\left(L_o - \mu_2 \frac{C_o^2}{U_o^2}\right) \\
e(y) &= -\frac{y}{D_o} \frac{C_o^2}{U_o^2} \\
\ell(y) &= -\frac{y}{D_o} p_3 L_o \\
m(y) &= -\frac{y}{D_o} p_2 \frac{C_o^2}{U_o^2} \\
n(y) &= -\frac{y}{D_o} q_2 \frac{C_o^2}{U_o^2} \, .
\end{aligned}
\tag{59}
$$

None of the functions $a(y)$, $b(y)$, $c(y)$ ... etc. is singular for $y \leq Y_o$.

The boundary conditions on G for $y = Y_o$ are, by (53) and (54) using (40),

$$G(x,Y_o) = 0, \quad G_y(x,Y_o) = p_1 H(x) + q_1 H_x(x) \tag{60}$$

and for $x = X_o$ from (53), (54) and (48)

$$G(X_o,y) = -\int_{y_o}^{y} \frac{U_{1p}}{U_o(1-U_o)}\, dy. \tag{61}$$

Two conditions are imposed on $H(x)$ through (51) and (52),

$$H(X_0) = \bar{R}_0^{\lambda-1}\left[\frac{\bar{t}_1}{\bar{t}_0} - \alpha\,\frac{\bar{R}_1}{\bar{R}_0}\right] \tag{62}$$

$$p_1 H_x(X_0) + q_1 H_x(X_0) = \left(\frac{dR_1}{dy}\right)_{y=Y_0} - \bar{R}_1.$$

Now the problem is reduced to solving a second order hyperbolic differential equation (58) for G as a function of x and y. It involves in the non-homogeneous term an unknown function $H(x)$. On the line $y = Y_0$, the two conditions (60) for G and G_y are imposed and one condition, (61) for G, on the line $x = X_0$. The quantity H and its first derivative are known at the point $x = X_0$, equations (51) and (52).

An indication that these data determine $G(x,y)$ and $H(x)$ is given by the following: If the "piston function" $G(X_0,y)$ can be represented by a power series around $y = Y_0$, then it is possible to determine a power series expansion for $G(x,y)$ about X_0, Y_0.

We wish to determine the behavior of G at the center, $x = \infty$. We would like, in fact, to determine $G(x,y)$ in the domain $X_0 \leq x < \infty$, $Y_\infty \leq y \leq Y_0$ where $y = Y_\infty$ corresponds to the position of Guderley's reflected shock at $\eta = H_\infty$. Beyond $y = Y_\infty$ we would need to fit a flow satisfying appropriate shock conditions at $y = Y_\infty$. This aspect of the problem is not discussed here since we are only concerned with finding out if the perturbation approximation breaks down as the flow approaches the center.

We examine now the flow picture in the (x,y)-plane. The differential equations of the characteristics are found most easily from equation (42):

$$D_o \left(\frac{dx}{dy}\right)^2 + \frac{2C_o^2}{U_o}\left(\frac{dx}{dy}\right) - \frac{C_o^2}{U_o^2} = 0$$

or

$$\frac{dx}{dy} = \frac{C_o}{D_o U_o}\,[C_o \pm (1-U_o)] = \frac{C_o}{U_o[C_o \mp (1-U_o)]}\,.$$

Thus the line $y = 0$ which corresponds to $(1 - U_o + C_o) = 0$ is a characteristic.

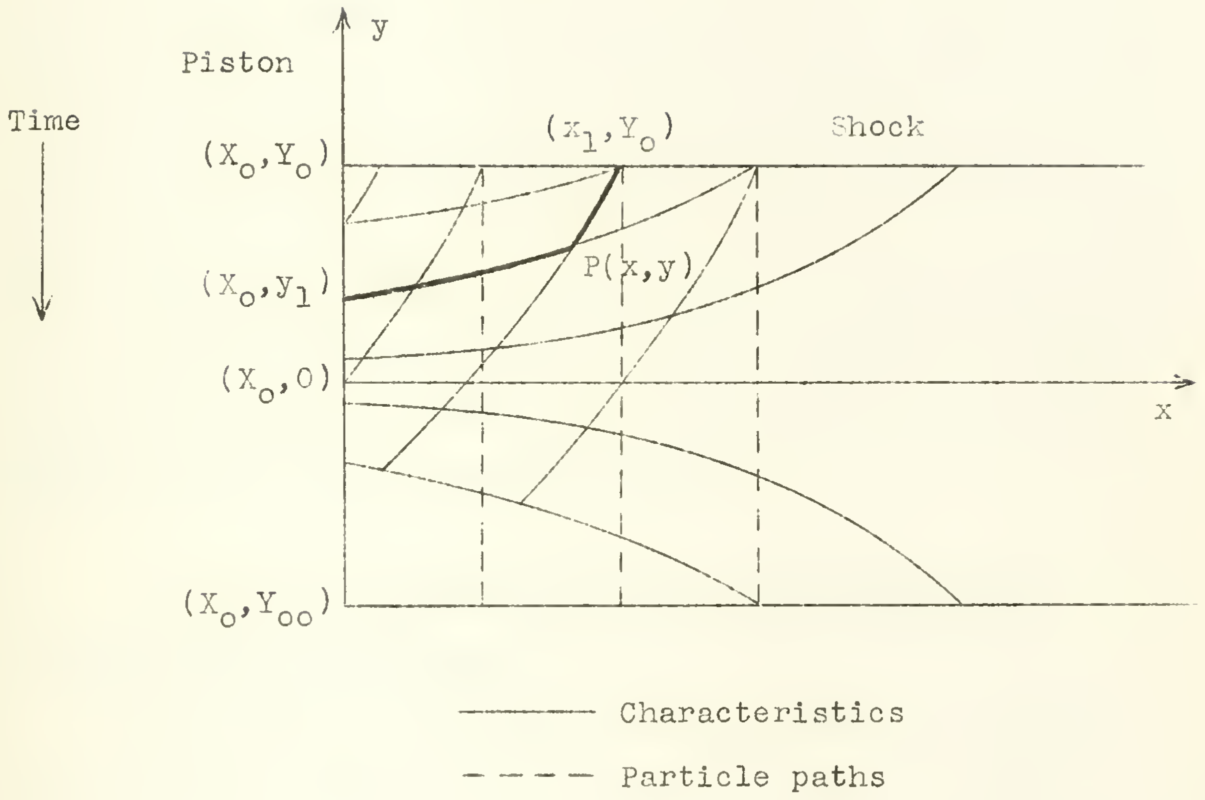
Piston
Time
y
(X_o, Y_o)
(x_1, Y_o)
Shock
(X_o, y_1)
$P(x, y)$
$(X_o, 0)$
x
(X_o, Y_{oo})
Characteristics
Particle paths

Figure 1.

From Figure 1 we see that since $y = 0$ is a characteristic, the value of G at any point P, with $X_o \leq x \leq \infty$, $0 \leq y \leq Y_o$, depends on the boundary values only in the range $X_o \leq x \leq x_1$ on $y = Y_o$ and $y_1 \leq y \leq Y_o$ on $x = X_o$ where $y_1 > 0$. Hence $H(x)$ for all values of x is determined by the piston motion up to $y = 0$. From the figure it is clear that any slight discontinuity in the piston motion near $y = 0$ might produce large effects for large values of x. We therefore assume that $G(X_o,y)$ is analytic for $y \leq Y_o$ and expect that a unique analytic solution exists. For the time being we consider only the segment $0 \leq y \leq Y_o$. For $y \leq 0$ we simply continue the flow analytically.

Part II. Application of the Laplace Transform

9. The Transformed Problem.

In the partial differential equation (58) for $G(x,y)$ the coefficients depend on y only and not on x. This suggests introducing the Laplace transform. The use of the Laplace transform is usually restricted to cases in which the normal derivative at $x = X_o$ is given as well as the function itself. However, the property that the lines $x = X_o$ and $y = Y_o$ are both timelike and that the equations are hyperbolic can be used to derive a relation between the normal derivative $G_x(X_o,y)$ and the given boundary conditions.

We assume a unique analytic solution $G(x,y)$, $H(x)$ of equations (58-62), Part I, exists such that

(1) $$g(y,k) = \int_{X_0}^{\infty} e^{-k(x-X_0)}\, G(x,y)\,dx$$

(2) $$h(k) = \int_{X_0}^{\infty} e^{-k(x-X_0)}\, H(x)\,dx$$

exist and converge absolutely for $\operatorname{Re} k \geq \kappa_0$ say.

If $G(x,y)$, $H(x)$ satisfy (58-62), Part I, then the transforms $g(y,k)$, $h(k)$ satisfy

(3) $$\begin{aligned} & y g'' + (a+bk)g' + (c+dk+ek^2)g \\ & \quad = (\ell + mk + nk^2)h(k) + P(y) + kQ(y) \end{aligned}$$

with

$$a(y) = \frac{y}{D_0} K_0$$

$$b(y) = -\frac{y}{D_0} \frac{2C_0^2}{U_0}$$

$$c(y) = \frac{y}{D_0} \mu_3 L_0$$

$$d(y) = -\frac{y}{D_0}\left(L_0 - \mu_2 \frac{C_0^2}{U_0^2}\right)$$

$$e(y) = -\frac{y}{D_0} \frac{C_0^2}{U_0^2}$$

$$\ell(y) = -\frac{y}{D_0} p_3 L_0$$

$$m(y) = -\frac{y}{D_0} p_2 \frac{C_0^2}{U_0^2}$$

$$n(y) = -\frac{y}{D_0} q_2 \frac{C_0^2}{U_0^2}\,.$$

Here

(4) $P(y) = bG_y(X_o,y) + dG(X_o,y) + eG_x(X_o,y) - mH(X_o) - nH_x(X_o)$

(5) $Q(y) = eG(X_o,y) - nH(X_o)$.

$G_y(X_o,y)$, $G(X_o,y)$ are given by (60) and (61), Part I, and $H(X_o)$ and $H_x(X_o)$ by (62) while $G_x(X_o,y)$ will eventually be determined. We note here

(6) $$G(X_o,Y_o) = 0$$

by (60),and (61), Part I.

From (1) and (2) and the boundary conditions (61) we obtain

(7) $$g(Y_o,k) = 0$$

(8) $$g_y(Y_o,k) = (p_1 + kq_1)h(k) - q_1H(X_o).$$

Since $G(x,y)$ is **regular** in y we also have $g(y,k)$ analytic in y. Equation (3) has exactly one singular point $y = 0$, and thus $g(y,k)$ is **regular** for all $y \leq Y_o$ if

(9) g is **regular** at $y = 0$.

Since g is a solution of (3) it can be written with appropriate functions $A(k)$ and $B(k)$ as

(10) $$g(y,k) = A(k)g_1(y,k) + B(k)g_2(y,k)$$
$$+ h(k)g_3(y,k) + g_4(y,k)$$

in which $g_1(y,k)$, $g_2(y,k)$ are two independent solutions of

the homogeneous equation

$$(11) \qquad \mathcal{L}(g) = yg'' + (a+bk)g' + (c+dk+ek^2)g = 0$$

while $g_3(y,k)$ is any regular solution of the equation

$$(12) \qquad \mathcal{L}(g) = \ell + mk + nk^2$$

and $g_4(y,k)$ is any regular solution of

$$(13) \qquad \mathcal{L}(g) = P + kQ.$$

We specify now which solutions we consider for g_1, g_2, g_3 and g_4. The indicial equation of (11) for $y = 0$ is

$$\delta(\delta - 1) + (a_o + b_o k)\delta = 0$$

where $a_o = a(o)$, $b_o = b(o)$. Equation (11) therefore has one solution that is regular at $y = 0$, and one solution which behaves like $y^{-a_o-b_o k+1}$ as $y \rightarrow 0$ except when $-a_o-b_o k+1$ is an integer.

Here $g_1(y,k)$ is the regular solution of (11) with

$$(14) \qquad g_1(0,k) = 1$$

and $g_2(y,k)$ is the singular solution with

$$(15) \qquad g_2(y,k) = y^{-a_o-b_o k+1}(1 + O(y))$$

near $y = 0$. Then $g_1(y,k)$ is defined except when $a_o+b_o k$ is a negative integer while $g_2(y,k)$ is defined except when $a_o+b_o k$ is a positive integer.

We turn next to the solutions of (12). Since the non-

homogeneous term is regular in y the regular solutions of (12) form a one-parameter family. We fix $g_3(y,k)$ for different ranges of k by the additional conditions

(16)
$$g_3(0,k) = \tau_{13} \quad \text{for} \quad \operatorname{Re} k \geq 0$$
$$g_3(Y_o,k) = \tau_{23} \quad \text{for} \quad \operatorname{Re} k < 0.$$

From general theory the function $g_3(y,k)$ is defined except when $a_o + b_o k$ is a negative integer.

Similarly we might fix $g_4(y,k)$ by the condition

(17)
$$g_4(0,k) = \tau_{14} \quad \text{for} \quad \operatorname{Re} k \geq 0$$
$$g_4(Y_o,k) = \tau_{24} \quad \text{for} \quad \operatorname{Re} k < 0.$$

Thus $g_1(y,k)$, $g_3(y,k)$, $g_4(y,k)$ are all regular in y but $g_2(y,k)$ singular at $y = 0$, provided $a_o + b_o k$ is not an integer. Since g is analytic in y, we have

(18)
$$B(k) \equiv 0.$$

From (7) and (8) we can determine the other coefficients in (10).

(19)
$$A(k) = \frac{A_1(k)}{D(k)}$$

(20)
$$h(k) = \frac{h_1(k)}{D(k)}$$

where

(21)
$$A_1(k) = \bar{g}_4\bar{g}_{3y} - \bar{g}_3\bar{g}_{4y} - q_1\underset{*}{\bar{H}}(X_o)\bar{g}_3 - (p_1 + kq_1)\bar{g}_4$$

$$(22) \qquad h_1(k) = \bar{g}_1\bar{g}_{4y} - \bar{g}_4\bar{g}_{1y} + q_1 H(X_o)\bar{g}_1$$

$$(23) \qquad D(k) = \bar{g}_3\bar{g}_{1y} - \bar{g}_1\bar{g}_{3y} + (p_1 + kq_1)\bar{g}_1 \; .$$

Here the bar means evaluated at $y = Y_o$. Note that $D(k)$ is independent of the value of g_3 at $y = 0$, see (16).

Thus we have

$$(24) \; g(y,k) = \frac{A_1(k)}{D(k)} g_1(y,k) + \frac{h_1(k)}{D(k)} g_3(y,k) + g_4(y,k).$$

Since $g_2(y,k)$ drops out $g(y,k)$ is defined for all values of k unless $a_o + b_o k$ is a negative integer or if $D(k) = 0$.

At this stage, the transform $g(y,k)$ cannot be determined directly from the boundary conditions given for $G(x,y)$. For, the function $g_4(y,k)$ depends on $P(y)$, see (13), which in turn depends through (4) on the function $G_x(X_o,y)$. However, from the behavior of g for Re k large and positive we can determine a relationship between $G(X_o,y)$ and $G_x(X_o,y)$. It will then be possible to determine some properties of $G(x,y)$ from the properties of $g(y,k)$.

9. Asymptotic Behavior of $g(y,k)$.

We have, see (1),

$$g(y,k) = \int_{X_o}^{\infty} e^{-k(x-X_o)} G(x,y)\, dx.$$

For Re k large and positive the main contribution to the integrand of this expression comes from $x = X_o$. Making use of this fact, we find for Re $k \longrightarrow +\infty$,

$$(25)\qquad g(y,k) \sim \frac{1}{k}\left(G(X_0,y) + O\left(\frac{1}{k}\right)\right).$$

If we use expression (24) for $g(y,k)$ we should find the same asymptotic behavior for $g(y,k)$. With this goal in mind we determine the asymptotic behavior of $g_1(y,k)$, $g_3(y,k)$, $g_4(y,k)$. This is done purely formally here and the results are proved in [4].

Using the usual procedure for finding asymptotic expansions we set

$$(26)\qquad g_1 = e^{-kw}\,\psi_1(y,k)$$

and substitute in the differential equation $\mathcal{L}(g_1) = 0$, see (11). We obtain

$$(27)\qquad y\psi_1'' + (a + bk - 2w'yk)\psi_1' + (c + (d-yw'' - aw')k + (e - bw' + y(w')^2)k^2)\psi_1 = 0.$$

The function $w(y)$ is determined by setting the coefficient of k^2 equal to zero.

$$(28)\qquad yw'^2 - b(y)w' + e(y) = 0.$$

Since $g_1(y,k)$ is regular at $y = 0$, $w(y)$ is chosen regular, so that from (28) we have

$$(29)\qquad \frac{dw}{dy} = \frac{b(y) - \sqrt{b^2(y) - 4e(y)y}}{2y}.$$

Then the differential equation for $\psi_1(y,k)$ is

$$(30)\qquad y\psi_1'' + (a + b_1k)\psi_1' + (c + d_1k)\psi_1 = 0$$

where

$$b_1(y) = \sqrt{b^2(y) - 4e(y)y} = \frac{y}{D_o}\,\frac{2C_o}{U_o}\,(1 - U_o)$$

$$d_1(y) = d(y) - a(y)w'(y) - yw''(y),$$

by Part I (59)

From $g_1(0,k) = 1$, see (14), and (26) we have

$$w(0) = 0$$

(31)

$$\psi_1(0,k) = 1$$

and therefore

$$(32) \qquad w(y) = \int_0^y \frac{b(\sigma) - \sqrt{b^2(\sigma) - 4e(\sigma)\sigma}}{2\sigma}\, d\sigma \,.$$

Since $b(y) > 0$, $b^2(y) - 4e(y)y > 0$, $e(y) > 0$, (see Part I (59)), the function $w'(y)$ is always positive and different from zero in the range $0 \leq y \leq Y_o$. At $y = 0$, $\frac{dw}{dy} = - \frac{e_o}{b_o} < 0$. The function $w(y)$ is therefore monotonic increasing. The condition $b^2(y) - 4e(y)y > 0$ is exactly the condition that equations (58) for $G(x,y)$ is hyperbolic. The condition $b(y) > 0$ is connected with the fact that the two lines $x = X_o$ and $y = Y_o$ are both timelike. As we shall see later, the possibility of solving the problem by using the Laplace transform rests on the fact that $w(y)$ is monotonic increasing.

From equation (30), which contains no term in k^2, an asymptotic expansion for ψ_1 is found by setting

$$(33) \qquad \psi_1(y,k) \sim \psi_{10}(y) + \frac{\psi_{11}(y)}{k} + \frac{\psi_{12}(y)}{k^2} + \dots \,.$$

Differential equations for $\psi_{10}(y)$, $\psi_{11}(y)$, etc. are found by direct substitution:

(34)
$$b_1\psi_{10}' + d_1\psi_{10} = 0$$
$$b_1\psi_{1n}' + d_1\psi_{1n} = -y\psi_{1,n-1}'' - a\psi_{1,n-1}' - c\psi_{1,n-1} \quad .$$

The condition (31) yields initial conditions for $\psi_{1n}(y)$.

(35)
$$\psi_{10}(0) = 1$$
$$\psi_{1n}(0) = 0 \qquad n \geq 1.$$

Equations (34) and (35) are easily solved.

A formal expansion for $g_3(y,k)$, see (12), can be found in a similar way. We set

(36)
$$g_3(y,k) \sim \psi_{30}(y) + \frac{\psi_{31}(y)}{k} + \dots - A_3(k)g_1(y,k)$$

where $\psi_{30}(y)$, ..., $\psi_{3n}(y)$ are determined by straight substitution in (12), $\mathcal{L}(g_3) = \ell + mk + nk^2$, and $A_3(k)$ is left undetermined. We find

(37)
$$+ e(y)\psi_{30} = n$$
$$+b(y)\psi_{30}' + d(y)\psi_{30} + e(y)\psi_{31} = m$$
$$y\psi_{30}'' + a(y)\psi_{30}' + b(y)\psi_{31}' + c(y)\psi_{30} + d(y)\psi_{31} + e(y)\psi_{32} = 0$$
$$y\psi_{3,n-2}'' + a(y)\psi_{3,n-2}' + b(y)\psi_{3,n-1}' + c(y)\psi_{3,n-2}$$
$$+ d(y)\psi_{3,n-1} + e(y)\psi_{3,n} = 0.$$

Since $e(y) = -\frac{y}{D_o}\frac{C_o^2}{U_o^2}$ (see Part I, (59)) does not vanish on $0 \leq y \leq Y_o$ the functions $\psi_{30}(y)$, $\psi_{31}(y)$, etc.

are all determined. We now fix A_3 (or τ_{13} and τ_{23} in (16)) asymptotically by requiring

$$(38)\qquad \begin{aligned} &g_3(0,k) \sim 0 \qquad \text{for} \quad \operatorname{Re} k \geq 0 \\ &g_3(Y_o,k) \sim \left(\psi_{30}(Y_o) + \frac{\psi_{31}(Y_o)}{k} + \dots\right) \quad \text{for} \quad \operatorname{Re} k < 0. \end{aligned}$$

In a similar way we fix $g_4(y,k)$

$$(39)\qquad g_4(y,k) \sim \frac{1}{k}\left(\psi_{40}(y) + \frac{\psi_{41}(y)}{k} + \dots\right) \qquad \text{for} \quad \operatorname{Re} k \geq 0$$

where

$$(40)\qquad e(y)\psi_{40} = Q(y)$$

$$(41.1)\qquad b(y)\psi_{40}' + d(y)\psi_{40} + e(y)\psi_{41} = P(y)$$

$$(41.2)\qquad y\psi_{40}'' + a(y)\psi_{40}' + b(y)\psi_{41}' + c(y)\psi_{40} + d(y)\psi_{41} + e(y)\psi_{42} = 0$$

$$(41.n)\qquad y\psi_{4,n-2}'' + a(y)\psi_{4,n-2}' + b(y)\psi_{4,n-1}' + c(y)\psi_{4,n-2} + d(y)\psi_{4,n-1} + e(y)\psi_{4,n} = 0.$$

For $\operatorname{Re} k < 0$, $g_4(y,k)$ is fixed by a formula analogous to (38). Combining (26), (33), (36), (39) we find

$$(42)\qquad \begin{aligned} &g_1(y,k) \sim e^{-kw(y)}\left(\psi_{10}(y) + \frac{T_1(y,k)}{k}\right) \\ &g_3(y,k) \sim \psi_{30}(y) + \frac{T_3(y,k)}{k} \\ &g_4(y,k) \sim \frac{1}{k}\left(\psi_{40}(y) + \frac{T_4(y,k)}{k}\right) \end{aligned}$$

where

$$
\begin{aligned}
T_1(y,k) &= \psi_{11}(y) + \frac{\psi_{12}(y)}{k} + \dots \\
(43) \qquad T_3(y,k) &= \psi_{31}(y) + \frac{\psi_{32}(y)}{k} + \dots \\
T_4(y,k) &= \psi_{41}(y) + \frac{\psi_{42}(y)}{k} + \dots
\end{aligned}
$$

are all $O(1)$ with respect to k. It has been shown in [4] that T_1, T_3 and T_4 are all bounded for $|k|$ sufficiently large and $\operatorname{Re} k > k_p$ where k_p can be chosen arbitrarily but cannot be $-\infty$.

Substituting (42) in (19-21) we find the asymptotic expansions of the coefficients $A_1(k)$, $h_1(k)$, $D(k)$

$$
\begin{aligned}
D(k) &= \bar{g}_3\bar{g}_{1y} - \bar{g}_{3y}\bar{g}_1 + (p_1+kq_1)\bar{g}_1 \sim k(w'(Y_o)\psi_{30}(Y_o) \\
&\qquad + q_1)e^{-kw(Y_o)}\psi_{10}(Y_o) + e^{-kw(Y_o)}T_5(Y_o,k)
\end{aligned}
$$

$$
\begin{aligned}
(44) \quad h_1(k) &= \bar{g}_1\bar{g}_{4y} - \bar{g}_4\bar{g}_{1y} + q_1H(X_o)\bar{g}_1 \sim (w'(Y_o)\psi_{40}(Y_o) \\
&\qquad + q_1H(X_o))e^{-kw(Y_o)}\psi_{10}(Y_o) + \frac{1}{k}e^{-kw(Y_o)}T_6(Y_o,k)
\end{aligned}
$$

$$
\begin{aligned}
A_1(k) &= \bar{g}_4\bar{g}_{3y} - \bar{g}_3\bar{g}_{4y} - q_1H(X_o)\bar{g}_3 - (p_1+kq_1)\bar{g}_4 \\
&\sim -q_1H(X_o)\psi_{30} - q_1\psi_{40} + \frac{1}{k}T_7(Y_o,k)
\end{aligned}
$$

where T_5, T_6, T_7 are all bounded in the same sense as T_1, T_3 and T_4.

We note here in particular the expression for $T_7(Y_o,k)$

$$(45)\quad T_7(Y_o,k) = \frac{1}{k}\left(\psi_{40}(Y_o) + \frac{\psi_{41}(Y_o)}{k} + \ldots\right)\Bigg(\psi'_{30}(Y_o) + \frac{\psi'_{31}(Y_o)}{k} + \ldots - p_1 - kq_1\Bigg) \; \frac{1}{k}\Bigg(\psi'_{40}(Y_o) + \frac{\psi'_{41}(Y_o)}{k} \ldots - q_1 kH(X_o)\Bigg)\left(\psi_{30}(Y_o) + \frac{\psi_{31}(Y_o)}{k} + \ldots\right)$$

$$= \sum_{n=0}^{\infty} \frac{1}{k^n}\Bigg\{ \sum_{0}^{n-1} (\bar\psi_{4r}\bar\psi'_{3,n-r} - \bar\psi'_{4r}\bar\psi_{3,n-r} - p_1\bar\psi_{4,n-1} - q_1\bar\psi_{4,n} - q_1 H(X_o)\bar\psi_{3,n}\Bigg\} \;,$$

where the bar indicates evaluation at $y = Y_o$.

From (44) and (24) we obtain the asymptotic behavior of $g(y,k)$ for

$$g(y,k) \sim \frac{-q_1H(X_o)\bar\psi_{30} - q_1\bar\psi_{40} + \dfrac{T_7(Y_o,k)}{k}}{\bar\psi_{10}(-\bar w'\bar\psi_{30}+q_1) + \dfrac{T_5(Y_o,k)}{k}} \frac{1}{k}\left[\psi_{10} + \frac{T_1(y,k)}{k}\right] e^{k(\bar w - w)}$$

$$(46)\qquad + \frac{\bar\psi_{10}(\bar w'\bar\psi_{40} + q_1H(X_o)) + \dfrac{T_6(Y_o,k)}{k}}{\bar\psi_{10}(-\bar w'\bar\psi_{30}+q_1) + \dfrac{T_5(Y_o,k)}{k}} \frac{1}{k}\left[\psi_{30} + \frac{T_3(y,k)}{k}\right]$$

$$+ \frac{1}{k}\left[\psi_{40} + \frac{T_4(y,k)}{k}\right]$$

$$\sim \frac{\bar\psi_{10}(-q_1H(X_o)\bar\psi_{30} - q_1\bar\psi_{40}) + \dfrac{T_7(Y_o,k)}{k}}{\bar\psi_{10}(-\bar w'\bar\psi_{30} + q_1)} \frac{\psi_{10}}{k} e^{k(\bar w - w)}\left[1 + 0\left(\frac{1}{k}\right)\right]$$

$$+ \frac{\bar\psi_{10}(\bar w'\bar\psi_{40} + q_1 H(X_o)}{\bar\psi_{10}(-\bar w'\bar\psi_{30} + q_1)} \psi_{30}\left[1 + 0\left(\frac{1}{k}\right)\right] + \frac{1}{k}\psi_{40}\left(1 + 0\left(\frac{1}{k}\right)\right) \;.$$

From (40), (36) and (5) we have

$$(47)\qquad \psi_{40}(y) = Q(y)/e(y) = G(X_o,y) - \frac{n(y)}{e(y)} H(X_o)$$

$$= G(X_o,y) - \psi_{30}(y)\, H(X_o).$$

Using also $G(X_o,Y_o) = 0$, (6), the asymptotic expansion for $g(y,k)$ reduces to

$$(48)\qquad g(y,k) \sim \frac{T_7(Y_o,k)\psi_{10}}{(-\overline{w}'\overline{\psi}_{30}+q_1)} \frac{e^{k(\overline{w}-w)}}{k^2}\left[1+0(\tfrac{1}{k})\right]$$

$$+ \frac{G(X_o,y)}{k}\left[1+0(\tfrac{1}{k})\right].$$

The asymptotic behavior for $h(k)$ is determined similarly:

$$(49)\qquad h(k) \sim \frac{H(X_o)}{k}\left(1 + 0(\tfrac{1}{k})\right).$$

Since $w(y)$ increases monotonically we have $\overline{w} - w = w(Y_o) - w(y) \geq 0$ for all y, $0 \leq y \leq Y_o$, and hence, from (48), $g(y,k)$ increases exponentially with k for $\operatorname{Re} k > 0$. But we know from (25) that g dies away like $\frac{1}{k}$ as $k \longrightarrow +\infty$. It follows that each coefficient in $T_7(Y_o,k)$ must vanish. It can be proved, see [4], that the successive vanishing of the coefficients in $T_7(Y_o,k)$ yields a series of equations relating the derivatives $P(y)$ and $Q(y)$ at $y = y_o$ and these relations through (4) and (5), determine $G_x(X_o,y)$ in terms of $G(X_o,y)$.

For the present we simply state that $P(y)$ is such that

$$e^{k\bar{w}}T_7(Y_o,k) \sim 0 \tag{50}$$

for $\operatorname{Re} k \geq 0$.

From equation (48) we then have

$$g(y,k) \sim \frac{G(X_o,y)}{k}\left(1 + O(\tfrac{1}{k})\right) \qquad \text{for } |k| > \bar{k}_p \tag{51}$$

$\operatorname{Re} k > k_p$, $0 \leq y \leq Y_o$.

For $Y_{oo} \leq y \leq 0$ we find exactly the same behavior.

Part III. Asymptotic Behavior of the Flow Near the Center.

11. The Inverse of $g(y,k)$.

We have assumed that the two integrals,

$$g(y,k) = \int_{X_o}^{\infty} e^{-k(x-X_o)}G(x,y)dx$$

$$h(k) = \int_{X_o}^{\infty} e^{-k(x-X_o)}H(x)dx$$

converge absolutely for $\operatorname{Re} k \geq \kappa_o$. We assume the solution $G(x,y)$, $H(x)$ is continuous differentiable on $X_o \leq x < \infty$, $0 \leq y \leq Y_o$. By the general theory for the inverse of a Laplace transform, we have then:

$$\lim_{T\to\infty} \frac{1}{2\pi i}\int_{\kappa_o-iT}^{\kappa_o+iT} e^{k(x-X_o)}\, g(y,k)dk = G(x,y) \qquad x > X_o \tag{1}$$

$$\lim_{T\to\infty} \frac{1}{2\pi i}\int_{\kappa_o-iT}^{\kappa_o+iT} e^{k(x-X_o)}h(k)dk = H(x) \qquad x > X_o.$$

By shifting the path of integration in these integrals from $\operatorname{Re}(k) = \kappa_o$ to $\operatorname{Re} k = \kappa_1$ where $\kappa_1 < \kappa_o$, it is possible to determine the asymptotic behavior of $G(x,y)$ and $H(x)$ as $x \longrightarrow \infty$.

We first investigate the continuation of $g(y,k)$ and $h(k)$ for $\operatorname{Re} k < \kappa_o$. From (20-24), we see that $g(y,k)$ and $h(k)$ are defined in the left half k-plane except at (a) the singularities of $g_1(y,k)$, $g_3(y,k)$ and $g_4(y,k)$ and (b) the zeros of $D(k) = \bar{g}_3\bar{g}_{1y} - \bar{g}_1\bar{g}_{3y} + (p_1+kq_1)\bar{g}_1$.

We assume for the present that the singularities of $g(y,k)$ and $h(k)$ are all simple poles at the points $k = k_n$ with residues $\tilde{g}(y,k_n)$ and $\tilde{h}(k_n)$ respectively. We shall study them in greater detail later.

From Part II, equations (49) and (51), we have $|kh(k)|$ and $|kg(y,k)|$ bounded for $|k| > \bar{k}_p$ $\operatorname{Re} k > k_p$ where k_p may be chosen arbitrarily. By a well-known application of the Tauberian theorem, the behavior of $G(x,y)$ as $x \longrightarrow \infty$ is given by

$$G(x,y) \sim \sum_n e^{k_n(x-X_o)}\, \tilde{g}(y,k_n) \tag{2}$$

and that of $H(x)$ by

$$H(x) \sim \sum_n e^{k_n(x-X_o)}\, \tilde{h}(k_n). \tag{3}$$

Whether $G(x,y)$ and $H(x)$ die out as $x \longrightarrow \infty$ depends on

whether any of the quantities k_n have positive real parts.

We now investigate the singularities of $g(y,k)$ in order to determine these quantities k_n. Singularities in $g(y,k)$ may occur if $D(k) = 0$ or if $g_1(y,k)$, $g_3(y,k)$, $g_4(y,k)$ are singular. We will show first that the singularities of g_1, g_3, g_4 do not affect $g(y,k)$ and $h(k)$. Then we will examine the values of k for which $D(k) = 0$.

12. The Singularities of $g_1(y,k)$, $g_3(y,k)$, $g_4(y,k)$.

As was pointed out in Section (9), the functions $g_1(y,k)$, $g_3(y,k)$ and $g_4(y,k)$ are defined for all k unless $a_o + b_o k = q$ where $q = 0, -1, -2, \ldots,$ etc., to $k = \alpha_q = \frac{q-a_o}{b_o}$. α_q is always negative since $a_o > 0$, $b_o > 0$. It can be shown now that g_1, g_3, and g_4 all have simple poles at $k = \alpha_q$ and that the residues are all proportional to the singular solution $g_2(y,k)$. Thus for $k \longrightarrow \alpha_q$ where $a_o + b_o\alpha_q = q$, we have

$$(4)\quad \begin{aligned} g_1(y,k)(k-\alpha_q) &\sim j_{1q}g_2(y,\alpha_q) + (k-\alpha_q)\phi_1(y,k) \\ g_3(y,k)(k-\alpha_q) &\sim j_{3q}g_2(y,\alpha_q) + (k-\alpha_q)\phi_3(y,k) \\ g_4(y,k)(K-\alpha_q) &\sim j_{4q}g_2(y,\alpha_q) + (k-\alpha_q)\phi_4(y,k) \end{aligned}$$

where $\phi_1(y,k)$, $\phi_3(y,k)$, $\phi_4(y,k)$ are appropriate functions analytic in the neighborhood of $k = \alpha_q$ and j_{1q}, j_{3q} and j_{4q} are appropriate constants.

On substituting these expressions in $T(y,k)$ and using (4) once more, we find that $(k-\alpha_q)T(y,k)$ is analytic near $k = \alpha_q$. Thus both $T(y,k)$ and $D(k)$ have simple poles at $k = \alpha_q$. Accordingly, from (5), $g(y,k)$ is not singular at $k = \alpha_q$. Similarly we find that $h(k)$ is not singular at $k = \alpha_q$.

This means that the functions $g(y,k)$ and $h(k)$ can have singularities in the k-plane only for those values of k for which the denominator $D(k) = \bar{g}_3\bar{g}_{1y} - \bar{g}_1\bar{g}_{3y} + (p_1+kq_1)\bar{g}_1$ vanishes. Note that this expression is independent of the particular choice of g_3.

13. The Eigenvalues.

We proceed next to examine the roots of $D(k) = \bar{g}_3\bar{g}_{1y} - \bar{g}_1\bar{g}_{3y} + (p_1+kq_1)\bar{g}_1$. For large values of $|k|$ and $\text{Re } k > k_p$ we have, see (44) Part II,

$$D(k) \sim k\left(\bar{\Psi}_{10}(\bar{w}'\bar{\Psi}_{30} + q_1) + O(\tfrac{1}{k})\right)$$

so that $D(k)$ has no roots in any right half plane if $|k|$ is sufficiently large. In other words, $D(k)$ has at most a finite number of roots with $\text{Re } k \geq 0$.

Since any two regular solutions of $\mathcal{L}(g) = \ell + mk + nk^2$ differ only by a multiple of $g_1(y,k)$, it follows that

$$
\begin{aligned}
(9) \qquad D(k) &= \bar{g}_3\bar{g}_{1y} - \bar{g}_{3y}\bar{g}_1 + (p_1+kq_1)\bar{g}_1 \\
&= (\bar{g}^*\bar{g}_{1y} - \bar{g}^*_y\bar{g}_1) + (p_1+kq_1)\bar{g}_1 \\
&= (-\bar{g}^*_y + p_1 + kq_1)\bar{g}_1
\end{aligned}
$$

where $g^*(y,k)$ is the regular solution of $\mathcal{L}(g^*) = \ell + mk + nk^2$ for which $g^*(Y_o,k) = \overline{g}^* = 0$. Thus $D(k) = 0$ if $\overline{g}_y^* = p_1 + kq_1$.

In other words, $D(k) = 0$ for the eigenvalues k_n of the following problem:

$$\begin{aligned} &\mathcal{L}(g^*) = \ell + mk + nk^2 \\ &g^*(Y_o,k) = 0 \\ &g_y^*(Y_o,k) = p_1 + kq_1 \\ &g^*(y,k) \text{ regular at } y = 0. \end{aligned} \tag{10}$$

In order to determine the residue $\tilde{g}(y,k_n)$ of $g(y,k)$ at $k = k_n$, we determine $g(y,k)$ in terms of $g^*(y,k)$ instead of $g_3(y,k)$. From (20) - (24) of Part II, we find then,

$$g(y,k) = -\frac{\overline{g}_4}{\overline{g}_1}\, g_1 + \frac{\overline{g}_1\overline{g}_{4y} - \overline{g}_4\overline{g}_{1y} + q_1 H(X_o)\overline{g}_1}{\overline{g}_1[-\overline{g}_y^* + p_1 + kq_1]}\, g^* + g_4\,. \tag{11}$$

We assume that $\frac{\partial}{\partial k}(\overline{g}_y^*) \neq q_1$ for $k = k_n$. It follows then from (11) that the residue $\tilde{g}(y,k_n)$ of $g(y,k)$ at k_n is given by:

$$\begin{aligned} \tilde{g}(y,k_n) &= \frac{\overline{g}_1\overline{g}_{4y} - \overline{g}_4\overline{g}_{1y} + q_1 H(X_o)\overline{g}_1}{\overline{g}_1\left(-\frac{\partial}{\partial k}\overline{g}_y^* + q_1\right)}\, g^*(y,k_n) \\ &= \Gamma_n g^*(y,k_n). \end{aligned} \tag{12}$$

Similarly the residue $\tilde{h}(k_n)$ of $h(k)$ at $k = k_n$ is found to be

$$\tilde{h}(y,k_n) = \Gamma_n. \tag{13}$$

Two eigenvalues of k are easily determined, namely $k = 0$ and $k = \lambda$. We prove this first in the case $k = 0$.

From the equations for the basic flow, (14) Part I, we see that if $U_o(\eta)$, $C_o(\eta)$, $P_o(\eta)$ is a solution then $U_o(\eta(1+\varepsilon_o))$, $C_o(\eta(1+\varepsilon_o))$, $P_o(\eta(1+\varepsilon_o))$ is also a solution. Expanding U_o, C_o, P_o in powers of ε_o we find by (14), Part I,

$$\begin{aligned} U_o(\eta(1+\varepsilon_o)) &= U_o(\eta) + \varepsilon_o \eta U_o = U_o(\eta) + \varepsilon_o \frac{A_o}{D_o} \\ (14)\quad C_o(\eta(1+\varepsilon_o)) &= C_o(\eta) + \varepsilon_o\, C_o = C_o(\eta)\left(1+\varepsilon_o \frac{B_o}{D_o}\right) \\ P_o(\eta(1+\varepsilon_o)) &= P_o(\eta) + \varepsilon_o\, P_o = P_o(\eta)\left(1+\varepsilon_o \frac{E_o}{D_o}\right) . \end{aligned}$$

This corresponds to the special solution of (24 and (39), Part I, that is independent of r:

$$(15)\qquad \begin{aligned} U_1(\eta,r) &= {}_oU_1(\eta) = \frac{A_o}{D_o} \\ P_1(\eta,r) &= {}_oP_1(\eta) = \frac{B_o}{D_o} \\ C_1(\eta,r) &= {}_oC_1(\eta) = \frac{E_o}{D_o} \\ H_1(r) &= {}_oH = -1 . \end{aligned}$$

Corresponding to this solution, equations (58) and (60) Part I for $G(x,y)$ and $H(x)$ have the solution

$$(16)\quad G(x,y) = {}_oG(y) = -\frac{1}{\mu_2}\left[\gamma^{-1}\frac{E_o}{D_o} - \frac{A_o}{D_o(1-U_o)}\right]_{H_o}^{\eta(y)}$$

$$H(x) = {}_oH = -1.$$

Thus ${}_oG(y)$ satisfies, see (58), the equation

$$y\,{}_oG_{yy} + a\,{}_oG_y + c\,{}_oG = \mathcal{L}, \quad \text{with} \quad {}_oG(Y_o) = 0 \qquad {}_oG_y(Y_o) = p_1$$

that is ${}_{0}G(y)$ is a solution of (10) with $k = 0$.

That $k = \lambda$ is also an eigenvalue follows from the fact that if $u(r,t)$, $c(r,t)$, $p(r,t)$ is a solution of the equations for spherical flow (1) Part I, then $u(r,t+\varepsilon_1)$, $c(r,t+\varepsilon_1)$, $p(r,t+\varepsilon_1)$ is also a solution. This means in the present case that $u = \frac{\alpha r}{t+\varepsilon_1} U_0(r^{-\lambda}(t+\varepsilon_1))$, $c = \frac{\alpha r}{t+\varepsilon_1} C_0(r^{-\lambda}(t+\varepsilon_1))$, $p = \frac{\alpha^2 r^2}{(t+\varepsilon_1)^2} P_0(r^{-\lambda}(t+\varepsilon_1))$ is a solution of (1) Part I where $U_0(\eta)$, $C_0(\eta)$, $P_0(\eta)$ is Guderley's fundamental solution. Expanding in powers of ε_1 according to the procedures of section 4 we find that

$$U_1(r,\eta) = {}_{\lambda}U_1(r,\eta) = \left[\frac{A_0}{D_0} - U_0\right] r^{-\lambda}/\eta \tag{17}$$

$$C_1(r,\eta) = {}_{\lambda}C_1(r,\eta) = \left[\frac{B_0}{D_0} - 1\right] r^{-\lambda}/\eta$$

$$P_1(r,\eta) = {}_{\lambda}P_1(r,\eta) = \left[\frac{E_0}{D_0} - 2\right] r^{-\lambda}/\eta$$

is a solution of (24) and (39) Part I if we take $H_1(r) = -r^{-\lambda}/H_0$. This solution corresponds to

$$G = {}_{\lambda}G(x,y) = \frac{e^{\lambda(x-y)}}{\lambda-1}\left[\frac{\gamma-1}{2\eta}\left(\frac{E_0}{D_0} - 2\right) - \frac{\gamma}{\eta}\left(\frac{B_0}{D_0} - 1\right)\right]_{\eta=H_0}^{\eta=\eta(y)}$$

$$H = {}_{\lambda}H(x) = -\frac{e^{\lambda(x-Y_0)}}{H_0}$$

being a solution of (58) and (60) Part I for $G(x,y)$ and $H(x)$. Substituting (18) in (58) Part I we find that

$$(19)\qquad {}_{\lambda}g(y) = H_0\,\frac{e^{-\lambda(y-Y_0)}}{\lambda-1}\left[\frac{\gamma-1}{2\eta}\left(\frac{E_0}{D_0}-2\right)-\frac{\gamma}{\eta}\left(\frac{B_0}{D_0}-1\right)\right]_{\eta=H_0}^{\eta=\eta(y)}$$

satisfies $\mathcal{L}({}_{\lambda}g) = \ell + m\lambda + n\lambda^2$, ${}_{\lambda}g(Y_0) = 0$ and ${}_{\lambda}g_y(Y_0) = p_1 + \lambda q_1$. Thus ${}_{\lambda}g(y)$ is an eigenfunction and λ an eigenvalue of the system (10).

The other eigenvalues are not readily calculated. In appendix II a rather crude method for calculating k_n and $g^*(y,k_n)$ is described. It appears from these preliminary calculations that the only non-negative values of k are 0 and λ and that the highest negative value of k is approximately -2.

It can also be shown that $D(k)$ vanishes at least once between $\alpha_q = \frac{q-a_0}{b_0}$ and $\alpha_{q+1} = \frac{q+1-a_0}{b_0}$ if q is a sufficiently large negative integer so that an infinite number of eigenvalues exist.

14. Behavior of the Flow Quantities at the Center.

Substituting equations (13) and (14) in the asymptotic expressions (2) and (3) for $G(x,y)$ and $H(x)$, we obtain the asymptotic representations:

$$(20)\qquad G(x,y) \sim \sum_n e^{k_n(x-X_0)}\,\Gamma_{k_n}\,g^*(y,k_n)$$

$$H(x) \sim \sum_n e^{k_n(x-X_0)}\,\Gamma_{k_n}$$

for $0 \leq y \leq Y_o$. The same behaviour can also be shown for $Y_{oo} \leq y \leq 0$.

Using (14) in the expressions for V_1, V_2, and V_3, (equations (53), (55), (56) and (40) Part I) we obtain

$$(21) \qquad V_1(x,y) \sim \sum_n e^{k_n(x-X_o)} \Gamma_{k_n} g_y^*(y,k_n)$$

$$V_2(x,y) \sim \sum_n e^{k_n(x-X_o)} \Gamma_{k_n} [p_2+k_n q_2-(k_n-\mu_2)g^*(y,k_n)]$$

$$V_3(x,y) \sim \sum_n e^{k_n(x-X_o)} \Gamma_{k_n} [p_3-(k_n-\mu_3)g^*(y,k_n)]$$

or using (27) Part I and (29) Part I,

$$(22) \qquad U_1(\eta,r) \sim \sum_n r^{-k_n} \Gamma_{k_n} u_1^*(\eta,k_n)$$

$$P_1(\eta,r) \sim \sum_n r^{-k_n} \Gamma_{k_n} P_1^*(\eta,k_n)$$

$$C_1(\eta,r) \sim \sum_n r^{-k_n} \Gamma_{k_n} c_1^*(\eta,k_n)$$

where

$$(23) \qquad u_1^*(\eta,k_n) = e^{k_n(y-X_o)} g_y^*(y,k_n)$$

$$p_1^*(\eta,k_n) = e^{k_n(y-X_o)} (p_2+k_n q_2-(k_n-\mu_2)g^*(y,k_n))$$

$$c_1^*(\eta,k_n) = e^{k_n(y-X_o)} (p_3-(k_n-\mu_3)g^*(y,k_n)).$$

Since $k = 0$ and $k = \lambda$ are two of the values of k_n it would appear from (22) that the flow quantities U_1, P_1 and C_1 do not die out as $r \longrightarrow 0$. However, if we examine

the original formulation of the problem in sections 4 and 5, Part I, we see that we are always free to choose for any actual physical situation the two constants $\overline{R}_1$ and $\overline{t}_1$ (see (34) Part I) which fix $H(X_o)$ through (51) and $H_x(X_o)$ through (52). We recall that $\overline{R}_o + \varepsilon\overline{R}_1$ is the position from which the shock starts and $\overline{t}_o + \varepsilon\overline{t}_1$ is the time at which it starts.

We could set about proving that it is possible to choose $\overline{R}_1$ and $\overline{t}_1$ in such a way that Γ_o and Γ_λ in (22) vanish identically and hence that U_1, C_1 and P_1 all vanish as $r \longrightarrow 0$.

However, we proceed to eliminate these roots in a more specific fashion. From section 14, pages 53 and 54, we have $u = \frac{ar}{t+\varepsilon_1} U_o(r^{-\lambda}(t+\varepsilon_1))$, $p = \frac{ar}{t+\varepsilon_1} P_o(r^{-\lambda}(t+\varepsilon_1))$, $c = \frac{ar}{t+\varepsilon_1} C_o(r^{-\lambda}(t+\varepsilon_1))$. But $u = \frac{ar}{t} U_o(r^{-\lambda}t(1+\varepsilon_o))$, $p = \frac{ar}{t} P_o(r^{-\lambda}t(1+\varepsilon_o))$, $c = \frac{ar}{t} C_o(r^{-\lambda}t(1+\varepsilon_o))$ are two solutions of the equations of flow (1) Part I. Here ε_o and ε_1 are considered of the same order as ε, see section 4. Combining them we see that

$$(24) \qquad \begin{aligned} \overline{u}_1(r,t) &= \frac{ar}{t+\varepsilon_1} U_o\Big(r^{-\lambda}(t+\varepsilon_1)(1+\varepsilon_o)\Big) \quad , \\ \overline{c}_1(r,t) &= \frac{ar}{t+\varepsilon_1} C_o\Big(r^{-\lambda}(t+\varepsilon_1)(1+\varepsilon_o)\Big) \quad , \\ \overline{p}_1(r,t) &= \frac{ar}{t+\varepsilon_1} P_o\Big(r^{-\lambda}(t+\varepsilon_1)(1+\varepsilon_o)\Big) \end{aligned}$$

is a solution to (1) Part I. Expanding in powers of ε_o and ε_1 and using (15-18), (10) and (23) we obtain

$$(25)\quad \lambda \frac{t}{r}\,\bar{u}(r,t) = U_0(\eta) + \varepsilon_0 u^*(\eta,0) + \varepsilon_1 r^{-\lambda}\, u^*(\eta,\lambda)$$
$$+ \text{ lower order terms in } \varepsilon_0,\ \varepsilon_1$$

$$\lambda \frac{t}{r}\,\bar{c}(r,t) = C_0(\eta)[1 + \varepsilon_0 c^*(\eta,0) + \varepsilon_1 r^{-\lambda} c^*(\eta,\lambda)$$
$$+ \text{ lower order terms in } \varepsilon_0,\ \varepsilon_1]$$

$$\lambda \frac{t}{r}\,\bar{p}(r,t) = P_0(\eta)[1 + \varepsilon_0 p^*(\eta,0) + \varepsilon_1 r^{-\lambda} p^*(\eta,\lambda)$$
$$+ \text{ lower order terms in } \varepsilon_0,\ \varepsilon_1].$$

Now from (22) and equation (23) Part I we have

$$(26)\quad U(\eta,r) = U_0(\eta) + \varepsilon U_1(r,\eta) \sim U_0(\eta) + \varepsilon \sum r^{-k_n} \Gamma_{k_n} u^*(\eta,k_n)$$

$$C(\eta,r)/C_0(\eta) = 1 + \varepsilon C_1(r,\eta) \sim 1 + \varepsilon \sum r^{-k_n} \Gamma_{k_n} c^*(\eta,k_n)$$

$$P(\eta,r)/P_0(\eta) = 1 + \varepsilon P_1(r,\eta) \sim 1 + \varepsilon \sum r^{-k_n} \Gamma_{k_n} p^*(\eta,k_n).$$

If we choose $\varepsilon_0 = \varepsilon\Gamma_0$ and $\varepsilon_1 = \varepsilon\Gamma_\lambda$, we obtain, using (24-26) and $\eta = r^{-\lambda}t$,

$$(27)\quad U(\eta,r) \sim \frac{1}{1+\varepsilon_1\eta^{-1}r^{-\lambda}}\, U_0\Big((1+\varepsilon_0)(\eta+\varepsilon_1 r^{-\lambda})\Big) + \varepsilon\sum r^{-k_n}\Gamma_{k_n} u^*(\eta,k_n)$$

$$C(\eta,r) \sim \frac{1}{1+\varepsilon_1\eta^{-1}r^{-\lambda}}\, C_0\Big((1+\varepsilon_0)(\eta+\varepsilon_1 r^{-\lambda})\Big) + \varepsilon\sum r^{-k_n}\Gamma_{k_n} c^*(\eta,k_n)$$

$$P(\eta,r) \sim \frac{1}{1+\varepsilon_1\eta^{-1}r^{-\lambda}}\, P_0\Big((1+\varepsilon_0)(\eta+\varepsilon_1 r^{-\lambda})\Big) + \varepsilon\sum r^{-k_n}\Gamma_{k_n} p^*(\eta,k_n).$$

Here k_n no longer assumes the values 0 and λ.

It is clear then that $U(\eta,r)$, $C(\eta,r)$ and $P(\eta,r)$ differ less and less from one of Guderley's solutions as $r \longrightarrow 0$ since the values of k_n are all negative.

Appendix. Computation of the Eigenvalues.

In section 13 we required the eigenvalues of the following non-homogeneous system: $g(y,k)$ is a function which satisfies the non-homogeneous equation:

$$(1) \qquad \mathcal{L}(g) = yg'' + (a+bk)g' + (c+dk+ek^2)g = \ell+mk+nk^2$$

see (16) Part III. The coefficients $a, b, c, d, e, \ell, m, n$ are all analytic functions of y.

The initial conditions satisfied by $g(y,k)$ are

$$(2) \qquad g'(Y_o,k) = p_1 + kq_1$$

$$(3) \qquad g(Y_o,k) = 0.$$

In addition, for special values of k, $g(y,k)$ is regular at $y = 0$. These special values of k are the eigenvalues we seek. We would like to approximate them numerically and show that there are no eigenvalues of k with positive real part except $k = \lambda = 1.3947$.

We reduce the problem to one which is easier to handle numerically. Suppose that k is one of the eigenvalues. Setting $g(y,k) = g_1(y,k)\psi(y,k)$ where $g_1(y,k)$ is the regular solution of $\mathcal{L}(g_1) = 0$ and $g_1(0,k) = 1$, see (11) and (14) Part II, we find using (1-3), the differential equation for $\psi(y,k)$

$$(4) \qquad yg_1\psi'' + (2yg_1' + (a+bk)g_1)\psi' = \ell + mk + nk^2$$

and the initial conditions

(5) $$\psi(Y_o,k) = 0 \qquad \text{at} \quad y = Y_o$$

(6) $$g_1'\psi + g_1\psi' = p_1 + kq_1 \qquad \text{at} \quad y = Y_o$$

or using (5)

(7) $$g_1\psi' = p_1 + kq_1.$$

From the condition that $g(y,k)$ and $g_1(y,k)$ are regular at $y = 0$, we see that $\psi(y,k)$ and hence $\psi'(y,k)$ are regular at $y = 0$. Setting

(8) $$\overline{\psi}(y,k) = g_1^2 y^{a_o+b_ok} \exp\int_0^y \frac{a-a_o+(b-b_o)k}{y}\,dy$$

we find using (4) and (7),

(9) $$\overline{\psi}(y,k)\psi'(y,k) = \int_{Y_o}^y y_1^{-1}\,\overline{\psi}(y_1,k)g_1^{-1}(y_1,k)(\ell(y_1) + m(y_1)k + n(y_1)k^2)dy_1 + (p_1+kq_1)g_1^{-1}(Y_o,k)\overline{\psi}(Y_o,k)$$

or

(10) $$\psi'(y,k) = \overline{\psi}^{-1}(y,k)\int_0^y y_1^{-1}\overline{\psi}(y_1,k)g_1^{-1}(y_1,k)(\ell(y_1) + m(y_1)k + n(y_1)k^2)dy_1 + C(k)\overline{\psi}^{-1}(y,k)$$

where

(11) $$C(k) = -\int_0^{Y_o} y_1^{-1}\overline{\psi}(y_1,k)g_1^{-1}(y_1,k)(\ell(y_1)+m(y_1)k+n(y_1)k^2)dy_1 + (p_1+kq_1)g_1^{-1}(Y_o,k)\overline{\psi}(Y_o,k).$$

Using (8) in the expression (10) for $\psi'(y,k)$ we see that $\psi'(y,k)$ is regular at $y = 0$ only if $C(k)$ vanishes, that is from (11),

$$(12)\quad (p_1+kq_1)g_1^{-1}(Y_o,k)\overline{\psi}(Y_o,k) = \int_0^{Y_o} y^{-1}\overline{\psi}(y,k)g_1^{-1}(y,k)$$

$$(\ell(y_1) + m(y_1)k + n(y_1)k^2)dy_1.$$

Equation (12) is therefore a condition on the eigenvalues of k.

Setting

$$(13)\qquad g_1^{-1}(y,k)\overline{\psi}(y,k) = y^{a_o+b_ok}F(y,k)$$

we rewrite (12) as

$$(14)\quad (p_1+kq_1)Y_o^{a_o+b_ok}\,F(Y_o,k) = \int_0^{Y_o} y^{a_o+b_ok-1}F(y,k)$$

$$(\ell(y)+m(y)k+n(y)k^2)dy.$$

From (8), (13) and $\mathcal{L}(g_1) = 0$ we obtain the differential equation for $F(y,k)$, namely

$$(15)\qquad yF'' + (a^*+b^*k)F' + (c^*+d^*k+e^*k^2)F = 0$$

where

$$a^*(y) = -a(y) + 2a(0)$$

$$b^*(y) = -b(y) + 2b(0)$$

$$c^*(y) = -(a(y)-a(0))y^{-1}(a(0)-1) - a'(y) + c(y)$$

$$d^*(y) = -(a(y)-a(0))y^{-1}b(0) - (b(y)-b(0))y^{-1}(a(0)-1)$$

$$-b'(y) + d(y)$$

$$e^*(y) = -(b(y)-b(0))y^{-1}b(0) + e(y).$$

From (8) and (13)

(16) $F(y,k)$ is regular at $y = 0$

and, using $g_1(0,k) = 1$

(17) $F(0,k) = 1.$

The method for calculating the values of k is now as follows. We expand $F(y,k)$ in increasing powers of y using (15-17):

(18) $$F(y,k) = 1 + F_1(k)y + F_2(k)y^2 + \dots .$$

We also expand $\ell(y) + m(y)k + n(y)k^2$ in powers of y:

(19) $$\ell(y) + m(y)k + n(y)k^2 = I_o + I_1(k)y + I_2(k)y^2 + \dots .$$

Substituting (18) and (19) in (14) and integrating we obtain,

(20) $$(p_1+kq_1)\sum_{n=0}^{\infty} F_n Y_o^n = \sum_{n=0}^{\infty} \frac{Y_o^n}{a_o+b_o k+n} \sum_{m=0}^{n} F_m I_{n-m} .$$

To find the first approximation for the first two eigenvalues we take just the first terms in the sums in (20). From $F_o = F(0,k) = 1$ and $I_o = \ell(0) + m(0)k + n(0)k^2$ we obtain

(21) $$p_1 + kq_1 = \frac{\ell(0)+m(0)k+n(0)k^2}{a_o+b_o k}$$

or

(22) $P^1(k) = (p_1+kq_1)(a_o+b_ok) - (\ell(0)+m(0)k+n(0)k^2) = 0.$

On substituting the appropriate numerical values we find

(23) $$P^1(k) = .1776 + .4683k + 3.3960k^2 = 0.$$

Solving (23) for k we obtain

$$k = -.037 \quad \text{or} \quad 1.416.$$

For the second approximation we use, see (15),

(24) $$F_1 = -\frac{c^*(0)+d^*(0)k+e^*(0)k^2}{a^*(0)+b^*(0)k} = -\frac{c^*(0)+d^*(0)k+e^*(0)k^2}{a_o+b_ok}$$

along with

(25) $$I_1 = \ell'(0) + m'(0)k + n'(0)k^2.$$

The relation corresponding to (20) can then be reduced to the polynomial

(26) $$P^2(k) = (a_o+b_ok)P^1(k) + y_1[(a'(0)+b'(0)k)(a_o+b_ok) - c_o - d_ok - e_ok^2][P^1(k) - y_1(p_1+kq_1)] + y_1(a_o+b_ok)(\ell'(0) + m'(0)k + n'(0)k^2) = 0$$

or substituting numerical values,

$$P^2(k) = .018250 + 5.10815k - 1.425139k^2 + 1.623692k^3 - .191186k^4 .$$

This method can be continued indefinitely. The approximation to (20) yields successively various rational relations

to be solved for k. Each rational relation reduces to a polynomial whose degree increases by two at each step.

Thus, as it stands, this method appears to yield two new eigenvalues at every step. The computations were made however by rejecting the higher order terms in k and dealing with polynomials which increase by only one degree in k at each step. Thus we reject the fourth order term in k in P^2, the fifth and higher terms in P^3 and so on. Each approximation then yields one new eigenvalue.

The third approximation comes then from the polynomial

$$P^3(k) = +.1226 + 10.2490k - .31369k^2 - 4.2308k^3 - .52936k^4$$

The approximate eigenvalues are given in Table I. The eigenvalues appear to be all negative except $k = 0$ and $k = \lambda = 1.395$. The first negative eigenvalue is approximately -2.

Table I.

	k_1	k_o	k_{-1}	k_{-2}
$P^1(k)$	1.416	-.037		
$P^2(k)$	1.390	-.0087	-2.3	
$P^3(k)$	1.410	-.012	-1.8	-7.6
Correct value	1.395	0	-	-

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